**UNIVERSITY OF SWAZILAND**

**FACULTY OF SCIENCE**

**DEPARTMENT OF ELECTRONIC ENGINEERING**

**MAIN EXAMINATION 2006**

**TITLE OF PAPER : MATHEMATICAL METHODS II ( PAPER ONE )**

**COURSE NUMBER : E470(I)**

**TIME ALLOWED : THREE HOURS**

**INSTRUCTIONS: ANSWER ANY FOUR OUT OF FIVE QUESTIONS.**

**EACH QUESTION CARRIES 25 MARKS.**

**MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.**

**THIS PAPER HAS SEVEN PAGES, INCLUDING THIS PAGE.**

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## E470(I) MATHEMATICAL METHODS II (PAPER ONE)

**Question one**

Given the following complex function $f(z) = 6e^{-2z} - 5z^3$ where $z = x + iy$ ,

(a) (i) find its $u(x,y)$ *and* $v(x,y)$ , **( 3 marks )**

(ii) check for its analyticity , **( 5 marks )**

(b) (i) plot the mapped image of a rectangular region in $z-plane$ defined by

$1 < x < 5$ *and* $-3 < y < 0$ onto $f-plane$ , **( 3 marks )**

(ii) plot the mapped image of a ring region in $z-plane$ defined by

$1 \le r \le 2$ *and* $0 \le \theta \le 2\pi$ onto $f-plane$ , **( 3 marks )**

(c) Evaluate the value of $\int_{z_1,l}^{z_2} f(z)\,dz$ , if $z_1 = 1 - 2i$ and $z_2 = -3 + 4i$ , and

(i) if $l$ is the straight line from $z_1$ *to* $z_2$ , **( 7 marks )**

(ii) use int command directly irrespective of the integration path $l$ ,then compare this result with that obtained in (i) and make a brief remark.

**( 4 marks )**

## Question two

(a) Determine the value of $a$ such that $u(x,y) = 7x^2 + a\,y^2 - 2x$ is a harmonic and then find its conjugate harmonic $v(x,y)$ . **( 6 marks )**

(b) Given the following complex function $f(z)$ as :

$$f(z) = \frac{6z + 12}{z^2 - 2z + 10}$$

(i) find the two roots of the denominator of $f(z)$ , i.e., $z_1$ and $z_2$ . Replace the denominator of $f(z)$ by $(z - z_1)(z - z_2)$ and then convert $f(z)$ into its partial fraction, **( 3 marks )**

(Hint : can use $roots(z^2 - 2z + 10, I)$ to find its complex roots)

(ii) find its convergent series representation of $f(z)$ about the expansion centre $z = 9 - 3i$ for all the values of $z$ in the region of $8 < |z - 9 + 3i| < 10$

**( 8 marks )**

(iii) evaluate the value of $\oint_l f(z)\,dz$ if $l$ : $|z - 3i| = 5$ and in counter clockwise sense , **( 3 marks )**

(c) Find the centre and the radius of convergence of the following power series :

$$\sum_{n=0}^{\infty} \frac{(-1)^n (4n)!}{5^n (2n)! (n!)^2} (z + 6 + i)^n$$ **( 5 marks )**

## Question two

(a) Determine the value of $a$ such that $u(x,y) = 7x^2 + ay^2 - 2x$ is a harmonic and then find its conjugate harmonic $v(x,y)$. **( 6 marks )**

(b) Given the following complex function $f(z)$ as :

$$f(z) = \frac{6z + 12}{z^2 - 2z + 10}$$

(i) find the two roots of the denominator of $f(z)$, i.e., $z_1$ and $z_2$. Replace the denominator of $f(z)$ by $(z - z_1)(z - z_2)$ and then convert $f(z)$ into its partial fraction, **( 3 marks )**

(Hint : can use $roots(z^2 - 2z + 10, I)$ to find its complex roots)

(ii) find its convergent series representation of $f(z)$ about the expansion centre $z = 9 - 3i$ for all the values of $z$ in the region of $8 < |z - 9 + 3i| < 10$

**( 8 marks )**

(iii) evaluate the value of $\oint_l f(z)\,dz$ if $l$ : $|z - 3i| = 5$ and in counter clockwise sense, **( 3 marks )**

(c) Find the centre and the radius of convergence of the following power series :

$$\sum_{n=0}^{\infty} \frac{(-1)^n (4n)!}{5^n (2n)! (n!)^2} (z + 6 + i)^n$$ **( 5 marks )**

**Question three**

(a) Given the following definite integral :

$$\int_0^{2\pi} \frac{\sin(2\theta)}{5 - 3\cos(\theta) + 2\sin(\theta)}\, d\theta$$

(i) use int command to find its value , **( 3 marks )**

(ii) convert it to a complex contour integral , evaluate the value of this contour integral . Compare it with that obtained in (i) . **( 9 marks )**

(b) Find the Cauchy principal value of the following integral :

$$\int_{-\infty}^{\infty} \frac{x - 6}{x^4 + 3x^3 + 4x^2 - 3x - 5}\, dx$$ **( 10 marks )**

Question four

(a) Given the following improper integral :

$$\int_{-\infty}^{\infty} \frac{x - 4}{x^4 - x^3 + 6x^2 - 7x + 15} \, dx$$

(i) use int command to find its value , **( 3 marks )**

(ii) convert it to a complex contour integral , evaluate the value of this contour integral . Compare it with that obtained in (i) . **( 7 marks )**

(b) Given the following improper integral :

$$\int_{-\infty}^{\infty} \frac{\cos(k\,x)}{x^6 + 9\,x^4 + 23\,x^2 + 15} \, dx \qquad \text{and} \qquad k > 0$$

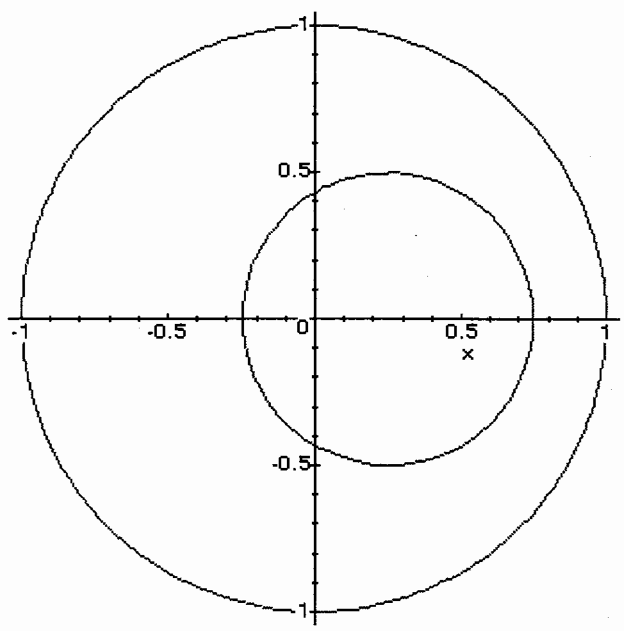
(i) convert it to a complex contour integral , find the result of this contour integral in terms of $k$ , **( 10 marks )**

(ii) evaluate the values of the given integrals when $k = 1.4$ . **( 2 marks )**

(iii) use int command to find its value of the given improper integral when $k = 1.4$ and compare it with that obtained in (ii) . **( 3 marks )**

**Question five**

A pair of long , non-coaxial , circular cross-section conductors is statically charged such that the inner conductor ( radius of $\frac{1}{2}$ and centred at $\left(x = \frac{1}{4}, y = 0\right)$ ) is at zero potential , i.e., $\Phi = 0$ volt , while the outer conductor ( radius of $1$ and centred at origin ) is maintained at $\Phi = 40$ volts as shown in the diagram below :




Use the linear fractional transformation of the form $w = \dfrac{z - b}{bz - 1}$ to transform the above given non-coaxial circles in $z-plane$ $(z = x + i\,y)$ to two coaxial circles in $w-plane$ $(w = u + i\,v)$ ,

(a) show that $w = \dfrac{z - b}{bz - 1}$ maps the unit circle in $z-plane$ onto the unit circle in $w-plane$ for any real value of $b$ , **( 5 marks )**

**Question five (continued)**

(b) find the appropriate value of $b$ such that the inner circle of radius $\frac{1}{2}$ maps to a coaxial circle of radius $r_0\,(< 1)$. Find also the value of $r_0$. **( 10 marks )**

(c) since the general solution for coaxial conductors can be written as $\Phi = k_1 \ln(|w|) + k_2$, determine the values of $k_1$ and $k_2$ from the given boundary conditions. **( 5 marks )**

(d) plot the equal potential surfaces $\Phi = 0$ , $\Phi = 10$ , $\Phi = 20$ , $\Phi = 30$ and $\Phi = 40$ in $z-plane$ and show them in a single display. **( 5 marks )**